

DYNAMICS AND STATICS OF VORTICES ON A PLANE AND A SPHERE — I

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Abstract

In the present paper a description of a problem of point vortices on a plane and a sphere in the “internal” variables is discussed. The Hamiltonian equations of motion of vortices on a plane are built on the Lie–Poisson algebras, and in the case of vortices on a sphere on the quadratic Jacobi algebras. The last ones are obtained by deformation of the corresponding linear algebras. Some partial solutions of the systems of three and four vortices are considered. Stationary and static vortex configurations are found.

1 Dynamics of point vortices on a plane

Introduction

Let us consider the plane motion in boundless ideal liquid of the N rectilinear filaments with the strengths Γ_i , crossing the plane in the points with coordinates (x_i, y_i) . It was shown by Kirchhoff [1] that the motion equations of such system can be written in the Hamiltonian form:

$$\Gamma_i \dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \Gamma_i \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad 1 \leq i \leq n, \quad (1)$$

with the Hamiltonian

$$H = -\frac{1}{8\pi} \sum_{i,j=1}^N \Gamma_i \Gamma_j \ln M_{ij}, \quad (2)$$

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$$M_{ij} \equiv (x_i - x_j)^2 + (y_i - y_j)^2,$$

The Poisson brackets are of the form:

$$\{f, g\} = \sum_{i=1}^N \frac{1}{\Gamma_i} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right). \quad (3)$$

The system of equations (1) has, beside of energy (2), the first integrals due to the invariance of the Hamiltonian with respect to translations and rotations of the coordinate system:

$$Q = \sum_{i=1}^N \Gamma_i x_i, \quad P = \sum_{i=1}^N \Gamma_i y_i, \quad I = \sum_{i=1}^N \Gamma_i (x_i^2 + y_i^2). \quad (4)$$

The set of integrals (4) is not in involution:

$$\{Q, P\} = \sum_{i=1}^N \Gamma_i, \quad \{P, I\} = -2Q, \quad \{Q, I\} = 2P. \quad (5)$$

Representation in “internal” variables

In this paper another representation of problem is given. Let us choose the squares of mutual distances between the point vortices as new variables. For systems of N vortices the number of mutual distances is equal to $N(N-1)/2$. Then, commuting M_{ij} and M_{lk} in the Poisson structure (3) and using the Heron formulas, we as a result obtain some nonlinear Poisson brackets in the phase space determined by mutual distances. However, if we add new variables, the oriented areas of parallelograms spanned on triples of vortices with numbers i, j, k :

$$\Delta_{ijk} \equiv (\mathbf{r}_j - \mathbf{r}_i) \wedge (\mathbf{r}_k - \mathbf{r}_i), \quad \mathbf{r}_l = (x_l, y_l), \quad (6)$$

then in the phase space of the variables $(M_{ij}, \Delta_{ijk}, i, j, k = 1, \dots, N)$ the Lie-Poisson brackets appear. The dimension of the phase is made up of the number of mutual distances between vortices (the number of binomial combinations from N on 2) and the number of triangles formed by the triples of vortices (the number of binomial combinations of N things \mathbf{r} at a time) (see Fig. 1): $C_N^2 + C_N^3 = C_{N+1}^3$.

Calculation of the Poisson brackets between these variables using (3) leads to the following relations:

$$\{M_{ij}, M_{kl}\} = 4 \left(\frac{1}{\Gamma_i} \delta_{ik} - \frac{1}{\Gamma_j} \delta_{jk} \right) \Delta_{ijl} + 4 \left(\frac{1}{\Gamma_i} \delta_{il} - \frac{1}{\Gamma_j} \delta_{jl} \right) \Delta_{ijk}; \quad (7)$$

$$\begin{aligned}
\{M_{ij}, \Delta_{klm}\} = & \left(\frac{1}{\Gamma_i} \delta_{ik} - \frac{1}{\Gamma_j} \delta_{jk} \right) (M_{li} - M_{im} + M_{mj} - M_{jl}) + \\
& + \left(\frac{1}{\Gamma_i} \delta_{il} - \frac{1}{\Gamma_j} \delta_{jl} \right) (M_{mi} - M_{ik} + M_{kj} - M_{jm}) + \\
& + \left(\frac{1}{\Gamma_i} \delta_{im} - \frac{1}{\Gamma_j} \delta_{jm} \right) (M_{ki} - M_{il} + M_{lj} - M_{jk});
\end{aligned} \tag{8}$$

$$\begin{aligned}
\{\Delta_{ijk}, \Delta_{lmn}\} = & \frac{\delta_{il}}{\Gamma_i} (\Delta_{jkn} - \Delta_{jkm}) + \frac{\delta_{im}}{\Gamma_i} (\Delta_{jkl} - \Delta_{jkn}) + \frac{\delta_{in}}{\Gamma_i} (\Delta_{jkm} - \Delta_{jkl}) + \\
& + \frac{\delta_{jl}}{\Gamma_j} (\Delta_{ikm} - \Delta_{ikn}) + \frac{\delta_{jm}}{\Gamma_j} (\Delta_{ikn} - \Delta_{ikl}) + \frac{\delta_{jn}}{\Gamma_j} (\Delta_{ikl} - \Delta_{ikm}) + \\
& + \frac{\delta_{kl}}{\Gamma_k} (\Delta_{ijn} - \Delta_{ijm}) + \frac{\delta_{km}}{\Gamma_k} (\Delta_{ijl} - \Delta_{ijn}) + \frac{\delta_{kn}}{\Gamma_k} (\Delta_{ijm} - \Delta_{ijl}).
\end{aligned} \tag{9}$$

Thus, in the C_{N+1}^3 -dimensional phase space of the variables (M, Δ) the Lie-Poisson brackets are determined. The structure of this algebra is as follows:

$$\{M, M\} \subset \Delta, \{M, \Delta\} \subset M, \{\Delta, \Delta\} \subset \Delta.$$

Besides the subalgebra Δ , this algebra contains subalgebras of problems of $(N-1)$, $(N-2), \dots, 3$ vortices.

The Poisson brackets are degenerated. The linear Casimir function

$$\Delta = \sum_{i,j=1}^N \Gamma_i \Gamma_j M_{ij} \tag{10}$$

is due to existence of the integral of angular momentum of vortex system. The linear Casimir functions correspond to the trivial geometric relations between the oriented areas of triangles. Let us write here, as an example, one such relation for a quadrangle $\Delta_{ijk} + \Delta_{ikl} - \Delta_{lij} - \Delta_{ljk} = 0$ (see Fig. 1). It is easy to see that the annihilator of the brackets is

$$F_{ijkl} = \Delta_{ijk} + \Delta_{ikl} - \Delta_{lij} - \Delta_{ljk} = 0. \tag{11}$$

The structure (9) has also the additional Casimir functions. They came about from such relations as

$$F_{ijk} = (2\Delta_{ijk})^2 + M_{ij}^2 + M_{jk}^2 + M_{ik}^2 - 2(M_{ij}M_{jk} + M_{ij}M_{ik} + M_{jk}M_{ik}) = 0 \tag{12}$$

derived from the Heron formula, relating the areas and lengths of sides of a triangle. In general even in the problem of four vortices the functions (12) are not Casimir

functions (they determine Casimir functions of subalgebra of three vortices). The detailed analysis and classification of Poisson brackets algebra (9) we plan to present in the next part of our work (to be published).

If we express Δ_{ijk} from (12) and substitute in the equations of motion for the squares of mutual distances M_{ij} , then we get the equations of Laura [2]. Having solved the equations of motion for the relative positions of vortices, we are able to find, using the quadratures and the initial conditions, the absolute coordinates of their positions on a plane at any time [3].

Partial cases of integrability of three and four vortices on a plane

The problems of three and four vortices are of particular interest. The integrability of the problem of three vortices was indicated already by Poincaré. The problem of four vortices is nonintegrable.

Under reduction of the ten-dimensional algebra, corresponding to the problem of four vortices on the general level of six Casimir functions, the Poisson structure becomes nondegenerate. So, according to the Darboux theorem, there exist local canonical coordinates, in which the equations have the form of a usual Hamiltonian system with two degrees of freedom. For integrability of such system, besides the Hamiltonian, one more first integral is required, which, in general case, does not exist.

Let us consider in detail a subalgebra (7)–(9), corresponding to the case of three vortices. In variables M , Δ it has the form:

$$\begin{aligned} \{M_1, M_2\} &= -\frac{4}{\Gamma_3}\Delta_4, \{M_3, M_1\} = -\frac{4}{\Gamma_2}\Delta_4, \{M_2, M_3\} = -\frac{4}{\Gamma_1}\Delta_4, \\ \{M_1, \Delta_4\} &= \left(\frac{1}{\Gamma_2} - \frac{1}{\Gamma_3}\right)M_1 + \left(\frac{1}{\Gamma_2} + \frac{1}{\Gamma_3}\right)(M_2 - M_3), \\ \{M_2, \Delta_4\} &= \left(\frac{1}{\Gamma_3} - \frac{1}{\Gamma_1}\right)M_2 + \left(\frac{1}{\Gamma_3} + \frac{1}{\Gamma_1}\right)(M_3 - M_1), \\ \{M_3, \Delta_4\} &= \left(\frac{1}{\Gamma_1} - \frac{1}{\Gamma_2}\right)M_3 + \left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2}\right)(M_1 - M_2). \end{aligned} \tag{13}$$

The rank of the algebra (13) is two. It corresponds on two-dimensional symplectic leaf to the system with one degree of freedom. Depending on the ratio of intensities, this algebra is a direct sum $SO(3) \oplus \mathbb{R}$ or $SO(1, 2) \oplus \mathbb{R}$ (the case $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ is under consideration). Analysis of this problem will be also given in our next publication.

One can point out at an interesting analogy between the problem of three vortices and the system of Lotka–Volterra, appearing in mathematical biology [4].

The Hamiltonian of three vortices

$$H = -\frac{1}{8\pi}(\Gamma_2\Gamma_3 \ln M_1 + \Gamma_1\Gamma_3 \ln M_2 + \Gamma_1\Gamma_2 \ln M_3) \quad (14)$$

generates the phase flow:

$$\hookrightarrow \dot{M}_1 = \{M_1, H\} = \frac{\Gamma_1}{2\pi} \left(\frac{1}{M_2} - \frac{1}{M_3} \right) \Delta_4, \quad (15)$$

$$\dot{\Delta}_4 = \frac{1}{8\pi} \left(\frac{\Gamma_2 + \Gamma_3}{M_1} (M_2 - M_3) + \frac{\Gamma_1 + \Gamma_3}{M_2} (M_3 - M_1) + \frac{\Gamma_1 + \Gamma_2}{M_3} (M_1 - M_2) \right).$$

Expressing Δ_4 from (12)

$$\Delta_4 = \pm \frac{1}{2} \sqrt{-M_1^2 - M_2^2 - M_3^2 + 2(M_1M_2 + M_1M_3 + M_2M_3)}, \quad (16)$$

we get the system:

$$\hookrightarrow \dot{M}_1 = \frac{\Gamma_1 \Delta_4}{2\pi M_1 M_2 M_3} M_1 (M_3 - M_2). \quad (17)$$

If we introduce the regularizing time τ :

$$\frac{d\tau}{dt} = \frac{\Delta_4}{2\pi M_1 M_2 M_3}, \quad (18)$$

then we find that M_i satisfy the Lotka–Volterra system:

$$\hookrightarrow \frac{d}{d\tau} M_1 = \Gamma_1 M_1 (M_3 - M_2). \quad (19)$$

Under transition of the system of vortices to a collinear configuration ($\Delta_4 = 0$) a change of time flow is possible, so one must choose in (16) the sign plus. Hence, we have demonstrated a piecewise trajectory isomorphism between two dynamical problems: the problem of three vortices and the three-dimensional Lotka–Volterra system, which takes place also in the problem of three vortices on a sphere, as it will be shown in the following section.

Let us note a partial case of integrability of four-vortex problem [5]:

$$\sum_{i=1}^4 \Gamma_i = 0, \quad Q = P = 0. \quad (20)$$

In this case, the system has sufficient number of integrals Q, P, I , in involution (see the algebra of integrals (5)). Its corresponding Casimir function is D on zero level: $D = 0$ due to the relation between the first integrals [6]:

$$D/2 = \left(\sum_{i=1}^N \Gamma_i \right) I - Q^2 - P^2. \quad (21)$$

Additional invariant relations which are necessary for integrability in variables M_{ij} one can find from the $N(N-1)$ identities [6]

$$f_{ij} \equiv \frac{1}{2} \sum_{k=1}^N \Gamma_k (M_{jk} - M_{ik}) = P(x_i - x_j) + Q(y_i - y_j), \quad (22)$$

being valid under $\sum_{i=1}^N \Gamma_i = 0$.

Let us indicate, for example, a particular solution of the system of four vortices if a central symmetry of the configuration takes place [3]. If $\Gamma_1 = \Gamma_3$, $\Gamma_2 = \Gamma_4$ and vortices form a parallelogram on a complex plane ($-z_1 = z_3$, $-z_2 = z_4$), then it preserves itself, i.e. in this case there are two invariant relations:

$$M_1 = L_1, \quad M_3 = L_3. \quad (23)$$

Having chosen an origin of coordinates, coinciding with the fixed symmetry center we can present the equations of motion in the form:

$$\begin{aligned} \frac{dM_1}{d\tau} &= \Gamma_1 M_1 L_2 (M_3 - M_2) + \Gamma_2 M_1 M_2 (L_2 - M_3), \\ \frac{dM_3}{d\tau} &= \Gamma_1 M_3 L_2 (M_2 - M_1) + \Gamma_2 M_2 M_3 (M_1 - L_2), \\ \frac{dM_2}{d\tau} &= 2\Gamma_2 M_2 L_2 (M_1 - M_3), \quad \frac{dL_2}{d\tau} = 2\Gamma_1 M_2 L_2 (M_3 - M_1). \end{aligned} \quad (24)$$

Here the notation: $L_2 \equiv M_{24}$ is used and, similarly to (18), the regularization of time τ :

$$\frac{d\tau}{dt} = \frac{\Delta_4}{2\pi M_1 M_2 M_3 L_2} \quad (25)$$

is made. The first pair of equations (24) describes motion of the sides of parallelogram and the second one describes motion of its diagonals.

Equations (24) are Hamiltonian with the Hamiltonian:

$$H = 2\Gamma_1 \Gamma_2 \ln M_1 + \Gamma_1^2 \ln M_2 + 2\Gamma_1 \Gamma_2 \ln M_3 + \Gamma_2^2 \ln L_2. \quad (26)$$

The Poisson brackets are quartic by elements of the algebra:

$$\begin{aligned} \{M_1, M_2\} &= \frac{1}{\Gamma_1} M_1 M_2 M_3 L_2, \\ \{M_1, M_3\} &= \frac{1}{2} \left(\frac{1}{\Gamma_1} - \frac{1}{\Gamma_2} \right) M_1 M_2 M_3 L_2, \\ \{M_2, M_3\} &= \frac{1}{\Gamma_1} M_1 M_2 M_3 L_2, \\ \{M_1, L_2\} &= -\frac{1}{\Gamma_2} M_1 M_2 M_3 L_2, \\ \{M_2, L_2\} &= 0, \quad \{M_3, L_2\} = \frac{1}{\Gamma_2} M_1 M_2 M_3 L_2. \end{aligned} \quad (27)$$

The Poisson structure can be received from the bracket (13) under the restriction on the mentioned above invariant relations. Now, its rank is two. The geometric Casimir function is obtained from the relation between the squares of sides and the squares of diagonals in the parallelogram:

$$F = 2(M_1 + M_3) - (M_2 + L_2). \quad (28)$$

For the real motions F is equal to zero.

The Casimir function (10) (“angular momentum” of the system) takes the form:

$$D = \Gamma_1 M_2 + \Gamma_2 L_2. \quad (29)$$

A reduction of system (24) on Casimir functions (28), (29) reduces it to the one-dimensional integrable Hamiltonian one [6].

One must note that in contrast to the rigid body dynamics the equations of motion in the “internal” variables do not have a standard invariant measure as well as even a measure with the analytical positive density. As a result dynamics of these problems is deeply different.

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2 Dynamics of point vortices on a sphere

Introduction

Let us consider a problem related to the previous one, a setting of which also goes back to the last century. In work [8] I. S. Gromeka obtained the equations of motion of point vortices on a sphere. The problem presents a simple model, describing a motion of vortex formations (cyclones etc.) in atmosphere of the Earth. The dynamical equations have been written in 1885 year yet were nearly forgot, though the problem in many respects is similar to the problem of motion of point vortices on a plane. In the papers [9, 10] there were pointed to integrability of three vortices on a sphere. In the papers [11, 12] there were proved the integrability of the restricted problem of four vortices.

We shall show that the equations of motion of N vortices on a sphere with the radius R can be presented in the Hamiltonian form with degenerated Poisson structure, assigned by Jacobi algebra $JQ(C_{N+1}^3)$ [13].

In fact, in the usual canonical basis the equations of motion of N vortices on a sphere can be written in the form:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad (30)$$

where the canonical coordinates q_k, p_k are expressed through the spherical angles θ_k, φ_k and the strengths Γ_k as

$$q_k = R\sqrt{\Gamma_k}\varphi_k, \quad p_k = R\sqrt{\Gamma_k}\cos\theta_k. \quad (31)$$

The Hamiltonian function has the form:

$$H = -\frac{1}{8\pi} \sum_{i,k=1}^N \Gamma_i \Gamma_k \ln(4R^2 \sin^2 \gamma_{ik}/2), \quad (32)$$

where

$$\cos \gamma_{ik} = \cos \theta_i \cos \theta_k + \sin \theta_i \sin \theta_k \cos(\varphi_i - \varphi_k) \quad (33)$$

is a cosine of the angle between i -th and k -th radius vectors, connecting the center of sphere with the point vortices (see Fig. 2).

The equations of motion of N point vortices in the spherical system of reference have the following form [9]:

$$\begin{aligned} \dot{\theta}_k &= -\frac{1}{4\pi R^2} \sum_{i=1}^N \Gamma_i \frac{\sin \theta_i \sin(\varphi_k - \varphi_i)}{1 - \cos \gamma_{ik}}, \\ \sin \theta_k \dot{\varphi}_k &= \frac{1}{4\pi R^2} \sum_{i=1}^N \Gamma_i \frac{\sin \theta_k \cos \theta_i - \cos \theta_k \sin \theta_i \cos(\varphi_k - \varphi_i)}{1 - \cos \gamma_{ik}}. \end{aligned} \quad (34)$$

They are Hamiltonian equations with Poisson brackets

$$\{\varphi_i, \cos \theta_k\} = \frac{\delta_{ik}}{R^2 \Gamma_i}. \quad (35)$$

The system (30) always has four integrals of motion

$$H = c_0,$$

$$F_1 \equiv \sum_i \Gamma_i \sin \theta_i \cos \varphi_i = c_1, \quad F_2 \equiv \sum_i \Gamma_i \sin \theta_i \sin \varphi_i = c_2, \quad F_3 \equiv \sum_i \Gamma_i \cos \theta_i = c_3, \quad (36)$$

which, however, are not in involution. It is possible to demonstrate that the integrals F_1, F_2, F_3 commute as components of angular momentum:

$$\{F_i, F_j\} = \varepsilon_{ijk} F_k. \quad (37)$$

Algebraic representation

Let us introduce the “internal” variables:

$$M_{ij} \equiv 4R^2 \sin^2 \gamma_{ij}/2, \quad (38)$$

being the squares of spans between arcs of corresponding vortices. The Hamiltonian (32) depends only on introduced mutual variables:

$$H = -\frac{1}{8\pi} \sum_{i,k=1}^N \Gamma_i \Gamma_k \ln M_{ik}. \quad (39)$$

From the relations between the canonical coordinates q_i, p_j (35) it is possible to find commutators between the values M_{ik} :

$$\{M_{ij}, M_{kl}\} = \frac{4}{R} \left(\frac{1}{\Gamma_i} \delta_{ik} - \frac{1}{\Gamma_j} \delta_{jk} \right) \Delta_{ijl} + \frac{4}{R} \left(\frac{1}{\Gamma_i} \delta_{il} - \frac{1}{\Gamma_j} \delta_{jl} \right) \Delta_{ijk}, \quad (40)$$

where the following notations are used:

$$\Delta_{ijk} \equiv \mathbf{r}_i \wedge \mathbf{r}_j \wedge \mathbf{r}_k. \quad (41)$$

Let us note that we use the same notations in both considered problems. Thus, the Poisson brackets between the values M_{ik} are proportional to external form of volume of parallelepiped spanned on the radius vectors of triple of vortices on a sphere.

It is possible to demonstrate that the algebra of variables M_{ik} and Δ_{ijk} is closed.

$$\begin{aligned} \{M_{ij}, \Delta_{klm}\} = & R \left(\frac{1}{\Gamma_i} \delta_{ik} - \frac{1}{\Gamma_j} \delta_{jk} \right) (M_{li} - M_{im} + M_{mj} - M_{jl}) + \\ & + R \left(\frac{1}{\Gamma_i} \delta_{il} - \frac{1}{\Gamma_j} \delta_{jl} \right) (M_{mi} - M_{ik} + M_{kj} - M_{jm}) + \\ & + R \left(\frac{1}{\Gamma_i} \delta_{im} - \frac{1}{\Gamma_j} \delta_{jm} \right) (M_{ki} - M_{il} + M_{lj} - M_{jk}) + \\ & + \frac{1}{2R} \left(\frac{1}{\Gamma_i} \delta_{ik} - \frac{1}{\Gamma_j} \delta_{jk} \right) (M_{jl} M_{im} - M_{mj} M_{il}) + \\ & + \frac{1}{2R} \left(\frac{1}{\Gamma_i} \delta_{il} - \frac{1}{\Gamma_j} \delta_{jl} \right) (M_{jm} M_{ik} - M_{jk} M_{im}) + \\ & + \frac{1}{2R} \left(\frac{1}{\Gamma_i} \delta_{im} - \frac{1}{\Gamma_j} \delta_{jm} \right) (M_{jk} M_{il} - M_{ik} M_{jl}); \end{aligned} \quad (42)$$

$$\begin{aligned}
\{\Delta_{ijk}, \Delta_{lmn}\} = & \frac{\delta_{il}}{\Gamma_i} \left(R(\Delta_{jkn} - \Delta_{jkm}) + \frac{1}{2R}(M_{in}\Delta_{jkm} - M_{im}\Delta_{jkn}) \right) + \\
& + \frac{\delta_{im}}{\Gamma_i} \left(R(\Delta_{jkl} - \Delta_{jkn}) + \frac{1}{2R}(M_{il}\Delta_{jkn} - M_{in}\Delta_{jkl}) \right) + \\
& + \frac{\delta_{in}}{\Gamma_i} \left(R(\Delta_{jkm} - \Delta_{jkl}) + \frac{1}{2R}(M_{im}\Delta_{jkl} - M_{il}\Delta_{jkm}) \right) + \\
& + \frac{\delta_{jl}}{\Gamma_j} \left(R(\Delta_{ikm} - \Delta_{ikn}) + \frac{1}{2R}(M_{jm}\Delta_{ikn} - M_{jn}\Delta_{ikm}) \right) + \\
& + \frac{\delta_{jm}}{\Gamma_j} \left(R(\Delta_{ikn} - \Delta_{ikl}) + \frac{1}{2R}(M_{jn}\Delta_{ikl} - M_{jl}\Delta_{ikn}) \right) + \\
& + \frac{\delta_{jn}}{\Gamma_j} \left(R(\Delta_{ikl} - \Delta_{ikm}) + \frac{1}{2R}(M_{jl}\Delta_{ikm} - M_{jm}\Delta_{ikl}) \right) + \\
& + \frac{\delta_{kl}}{\Gamma_k} \left(R(\Delta_{ijn} - \Delta_{ijm}) + \frac{1}{2R}(M_{kn}\Delta_{ijm} - M_{km}\Delta_{ijn}) \right) + \\
& + \frac{\delta_{km}}{\Gamma_k} \left(R(\Delta_{ijl} - \Delta_{ijn}) + \frac{1}{2R}(M_{kl}\Delta_{ijn} - M_{kn}\Delta_{ijl}) \right) + \\
& + \frac{\delta_{kn}}{\Gamma_k} \left(R(\Delta_{ijm} - \Delta_{ijl}) + \frac{1}{2R}(M_{km}\Delta_{ijl} - M_{kl}\Delta_{ijm}) \right). \tag{43}
\end{aligned}$$

The quadratic algebra of vortices on a sphere is presented as a deformation of the linear algebra of vortices on a plane using the parameter — the radius of curvature of sphere R . The algebra of vortices on a plane (7)–(9) and the corresponding Casimir functions of system on a sphere are obtained in the limit $R \rightarrow \infty : \Delta^v / R \rightarrow \Delta$, $M^v \rightarrow M$ from the algebra (40)–(43). Here we symbolically marked by a symbol v the variables of the problem on the sphere.

The linear Casimir functions of the quadratic algebra analogous to the corresponding functions of the linear algebra (10)–(11). The analog of the relations (12) can be written as follows:

$$\begin{aligned}
F_{ijk} = & 2\Delta_{ijk}^2 + R^2(M_{ij}^2 + M_{ik}^2 + M_{jk}^2) - \\
& - 2R^2(M_{ij}M_{jk} + M_{ik}M_{jk} + M_{ij}M_{ik}) + M_{ij}M_{jk}M_{ki} = 0. \tag{44}
\end{aligned}$$

Partial cases of integrability of three and four vortices on a sphere

In the spherical problem, the analogue of the partial case on the plane is a situation when the integrals of motion (36) have zero values:

$$F_1 = F_2 = F_3 = 0. \tag{45}$$

The case is completely integrable because there are sufficient number of commuting, on zero level, integrals.

It is not difficult to obtain identity, analogous to (21):

$$D/2 = \left(R \sum_{i=1}^N \Gamma_i \right)^2 - F_1^2 - F_2^2 - F_3^2. \quad (46)$$

Consequently, the considered case corresponds to the level of Casimir function:

$$D/2 = \left(R \sum_{i=1}^N \Gamma_i \right)^2.$$

In order to find the invariant relations in variables M_{ij} it is possible to use identity, between absolute and mutual coordinates, analogous to (22):

$$f_{ij} = \frac{1}{2} \sum_{k=1}^N \Gamma_k (M_{jk} - M_{ik}) = R(F_1(x_i - x_j) + F_2(y_i - y_j) + F_3(z_i - z_j)). \quad (47)$$

3 Stationary vortex states

The obtained forms of the dynamical equations on a plane and on a sphere can be successfully used for finding the steady-state configurations, being the partial solutions of the equations of motion.

Requirements of $N(N-1)/2$ mutual distances $\dot{M}_{ij} = 0$ being constant are the conditions of stationary state of vortex configurations. They have the same form both for the configurations on a plane and on a sphere in the “internal” variables as its Hamiltonians (2), (39) and commutation relations $\{M_{ij}, M_{kl}\}$ (7), (39) have the same form:

$$\sum_{l=1}^N \left(\frac{1}{M_{il}} - \frac{1}{M_{jl}} \right) \Gamma_l \Delta_{ijl}(M) = 0. \quad (48)$$

The conditions (48) are very suitable for finding symmetric configurations. On the plane there is solution of N vortices of equal strengths Γ , disposed in vertices of regular polygon, refined into a circle of radius R_0 . The system rotates with angular velocity

$$\Omega = \frac{\Gamma(N-1)}{4\pi R_0^2}. \quad (49)$$

The analysis of stability of such configurations in linear approximation was carried out in [15, 16].

For investigation of such configurations J. J. Thomson was awarded with Adams prize in 1883 [15]. These solutions were at the basis of the theory of vortex atoms propagandized by W. Kelvin (before quantum mechanics). Analogous configuration of identical vortices on a sphere is disposed, according to (52), on latitude $\theta = \theta_0$, coordinates φ are connected by conditions: $\varphi_k - \varphi_i = (k - i)2\pi / N$ (see Fig. 3). Angular velocity of rotation about the axis z is:

$$\Omega = \frac{\Gamma(N - 1)}{4\pi R_0^2} \cos \theta_0. \quad (50)$$

The angular velocity of the chain of vortices decreases from poles to equator and on the equator the configuration is static.

The next solution of the system (48) are collinear configurations: N identical vortices on a line rotate with some angular velocity around axis perpendicular to the plane. On the sphere of radius R the collinear configuration (N atoms on the meridian) rotates around the axis z with angular velocity Ω (see Fig. 4).

Collinear configurations of vortices on a sphere were not discussed in literature before. We can find them using the following conditions: coordinates φ_i of the point vortices have the equal values or differ on π . In addition: $\Delta_{ijl}(\mathbf{M}) = 0$. It is easy to notice that coordinates θ_i , ($0 \leq \theta \leq 2\pi$) are roots of the following system of trigonometric equations:

$$\frac{4\pi R^2 \Omega}{\Gamma} \sin \theta_k = \sum_{i=1}^N {}' \cot \left(\frac{\theta_k - \theta_i}{2} \right), \quad (1 \leq k \leq N). \quad (51)$$

The chain of vortices (51) deserves more attention. The roots of the system assigned the collinear configuration θ_i can be found as equilibrium position of the system of N particles on a circle set by the Hamiltonian

$$H = \frac{1}{2} \sum_{k=1}^N p_k^2 + \frac{4\pi R^2 \Omega}{\Gamma} \sum_{k=1}^N \cos \theta_k + \sum_{i, k=1}^N {}' \ln \left| \sin \frac{\theta_k - \theta_i}{2} \right|. \quad (52)$$

Evidently, the positions of equilibrium of the chain of particles coincide with roots of the system (51). Such connection in plane case was marked by Calogero [17]. The locations of identical vortices on a line are set by zeros of Hermitean polynomials of N -th power. The system (52) under $\Omega = 0$ was considered in [18], where statistical properties of levels of energies of one-dimensional classical Coulomb gas are considered. In the paper [19] the results of equilibrium analysis of the system (52) under condition $\Omega = 0 : \theta_i = \theta_0 + k\pi / N, k = 1, \dots, N$ are adduced.

The problem defined by the Hamiltonian (52) is usually nonintegrable.

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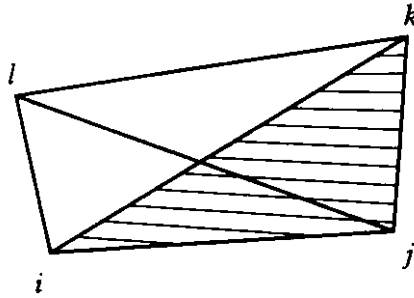


Figure 1:

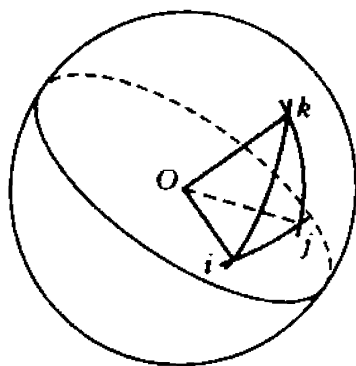


Figure 2:

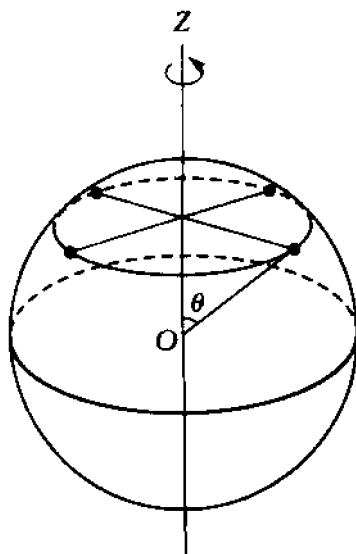


Figure 3:

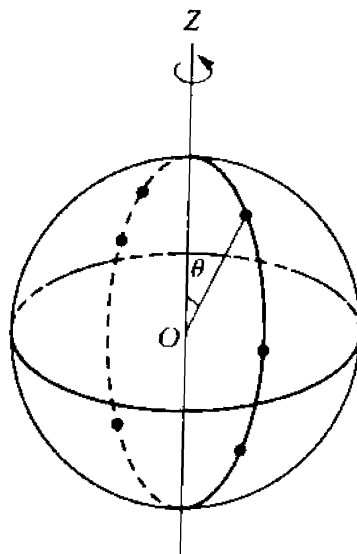


Figure 4: